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ERRATUM TO: "BANACH SPACES WITHOUT LOCAL UNCONDITIONAL STRUCTURE"

BY

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ABSTRACT

This note contains a corrected proof of the main result (which remains unchanged) from [K-T]. It was recently observed that an argument in a basic technical criterium has a gap.

The first named author recently observed that our paper [K-T] contains a gap. Here we shall correct the formulation of a general result (Theorem 2.1 in [K-T]) and then we shall complete the proof of the main application to the homogeneous Banach space problem (Theorem 3.1 in [K-T]), that is now neccessary in view of modifications in the aforementioned general statement. Corollaries 4.3, 4.4 and 4.5 from [K-T] do not require any changes.

This note is not self-contained. However, in order to make it possible for the readers a little familiar with [K-T] to read it without constantly referring to the original paper, we occasionally recall background information from [K-T].

The gap is in the proof of the fundamental criterium, Proposition 1.1 in [K-T]. One deals there with a Banach space Y which has a 2-dimensional

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Schauder decomposition $\{Z_k\}_k$. Assuming that Y has cotype $q < \infty$ and Gordon–Lewis l.u.st., one constructs a block diagonal operator T, which was claimed to be bounded on subspaces Y_K for which the decomposition $\{Z_k\}_{k \in K}$ was unconditional. In fact, we can estimate $\|T|_{Y_K}\|$ only if Y_K is complemented by the natural projection. To be more precise, for a subset $K \subset \mathbb{N}$ we define $Q_K: Y \to Y_K$ by $Q_K(\sum_k z_k) = \sum_{k \in K} z_k$. Then the correct formulation of Proposition 1.1 states, in condition (ii), that if the decomposition $\{Z_k\}_{k \in K}$ of Y_K is C-unconditional and $\|Q_K\| \leq C'$, then

$$\|T|_{Y_K}: Y_K \to Y_K\| \leq C^2 C' \psi \,\text{l.u.st.}(Y), \tag{1}$$

where ψ is the same function as in [K-T].

The proof uses the same operator T as in [K-T], and the estimate is completely clear from the original proof, once we notice that letting $P_k: Y \to Z_k$ denote the natural projection onto Z_k, for $k = 1, 2, \ldots$, we trivially have $P_k = P_k Q_K$ for every $k \in K$, and thus, for arbitrary signs $\varepsilon_k = \pm 1$, we have

$$\Big\|\sum_{k \in K} \varepsilon_k P_k: Y \to Y_K\Big\| = \Big\|\sum_{k \in K} \varepsilon_k P_k Q_K\Big\| \leq C\, C'.$$

Remark: Note that if the decomposition $\{Z_k\}_k$ of Y is unconditional, the natural projections Q_K are automatically bounded for all $K \subset \mathbb{N}$. In this case the proof of Proposition 1.1 in [K-T] was correct, so Corollaries 4.3, 4.4 and 4.5 do not require any changes.

The remaining part of this note depends heavily on technical notation from the end of Section 1 of [K-T]. For the reader's convenience we recall the most important one.

If $\Delta = \{A_m\}_m$ and $\Delta' = \{A'_m\}_m$ are partitions of $\mathbb{N}$, we say that $\Delta \succ \Delta'$, if there exists a partition $\mathcal{J}(\Delta', \Delta) = \{J_m\}_m$ of $\mathbb{N}$ such that

$$\min J_m < \min J_{m+1} \qquad \text{and} \qquad A'_m = \bigcup_{j \in J_m} A_j \qquad \text{for } m = 1, 2, \ldots. \tag{2}$$

If $\Delta_i = \{A_{i,m}\}_m$, for $i = 1, 2, \ldots$, is a sequence of partitions of $\mathbb{N}$, with $\Delta_1 \succ \cdots \succ \Delta_i \succ \cdots$, we set, for $m = 1, 2, \ldots$ and $i = 2, 3, \ldots$,

$$\mathcal{K}_{i,m} = \{K \subset A_{i,m} \mid |K \cap A_{i-1,j}| = 1 \text{ whenever } A_{i-1,j} \subset A_{i,m}\}. \tag{3}$$

Now Theorem 2.1 from [K-T] requires an obvious extra assumption of boundness of natural projections. Although the statement of this theorem is rather lengthy, we recall it here (in a corrected form) for the sake of future reference.

THEOREM ⋆2.1: *Let $X = F_1 \oplus \cdots \oplus F_4$ be a direct sum of Banach spaces of cotype r, for some $r < \infty$, and let $\{f_{i,l}\}_l$ be a normalized monotone Schauder basis in F_i, for $i = 1, \dots, 4$. Let $\Delta_1 \succ \cdots \succ \Delta_4$ be partitions of $\mathbb{N}$, $\Delta_i = \{A_{i,m}\}_m$ for $i = 1, \dots, 4$. Assume that there exist $C, C' \geq 1$ such that for every $K \in \mathcal{K}_{i,m}$ with $i = 2, 3, 4$ and $m = 1, 2, \dots$, the basis $\{f_{s,l}\}_{l \in K}$ in $F_s|_K$ is C-unconditional, and the natural projection $R_K^{(s)}: F_s \to F_s|_K$ has the norm $\|R_K^{(s)}\| \leq C'$, for $s = 1, \dots, 4$; moreover, there is $\widetilde{C} \geq 1$ such that for $i = 1, 2, 3$ and $m = 1, 2, \dots$ we have*

$$\|I: F_i|_{A_{i,m}} \to F_{i+1}|_{A_{i,m}}\| \leq \widetilde{C}. \tag{4}$$

Assume finally that one of the following conditions is satisfied:

(i) *there is a sequence $0 < \delta_m < 1$ with $\delta_m \downarrow 0$ such that for every $i = 1, 2, 3$ and $m = 1, 2, \dots$ and every $K \in \mathcal{K}_{i+1,m}$ we have*

$$\|I: D(F_1 \oplus \cdots \oplus F_i)|_K \to F_{i+1}|_K\| \geq \delta_m^{-1}; \tag{5}$$

(ii) *there is a sequence $0 < \delta_m < 1$ with $\delta_m \downarrow 0$ and $\sum_m \delta_m^{1/2} = \gamma < \infty$ such that for every $i = 1, 2, 3$ and $m = 1, 2, \dots$ and every $K \in \mathcal{K}_{i+1,m}$ we have*

$$\|I: F_{i+1}|_K \to F_i|_K\| \geq \delta_m^{-1}. \tag{6}$$

Then there exists a subspace Y of X without local unconditional structure, but which still admits a Schauder basis.

The only addition to the original proof is to check that the assumption for $\|Q_K: Y \to Y_K\|$ required in the fundamental criterion is now satisfied. If x_k and y_k are given by the same formulas as in [K-T], then an obvious calculation shows that

$$Q_K(z) = \sum_{s=1}^{4} R_K^{(s)} \Big(\sum_k (t_k \alpha_{s,k} + t'_k \alpha'_{s,k}) f_{s,k} \Big), \qquad \text{for } z = \sum_k (t_k x_k + t'_k y_k),$$

and thus $\|Q_K\| \leq 4 \max_s \|R_K^{(s)}\|$.

The remaining part of this note contains the proof of Theorem 3.1 from [K-T], which says:

THEOREM 3.1: *Let X be a Banach space with an unconditional basis and of cotype r, for some $r < \infty$. If X does not contain a subspace isomorphic to l_2 then there exists a subspace Y of X without local unconditional structure, which admits a Schauder basis.*

Let $\{e_l\}_l$ be a 1-unconditional basis in X. Passing to a subsequence we may assume without loss of generality that $\{e_l\}_l$ generates a spreading model $\{u_l\}_l$. We distinguish two main cases.

(I) Either no sequence of disjointly supported blocks of $\{e_l\}_l$ of length ≤ 3 with constant coefficients is equivalent to $\{u_l\}_l$,

(II) or the basis $\{e_l\}_l$ is 1-subsymmetric.

If Case I does not hold, then there exist disjointly supported blocks $\{w_l\}_l$ equivalent to $\{u_l\}_l$; so by renorming we get into Case II for $X_1 = \overline{\text{span}}\{w_l\}_l$. (Note that the length of these blocks is not used here.)

Case I is new, compared with [K-T]. It is contained in the following proposition proved by B. Maurey and N. Tomczak in May 1993 (unpublished).

PROPOSITION I: *Let X be a Banach space of cotype r, for some $r < \infty$. Assume that X has a 1-unconditional basis $\{e_l\}_l$ which generates a spreading model $\{u_l\}_l$. If no disjointly supported blocks of $\{e_l\}_l$ of length ≤ 3 with constant coefficients are equivalent to $\{u_l\}_l$, then there exists a subspace Y of X without local unconditional structure, which admits a 2-dimensional unconditional decomposition.*

Proof: Denote the span of $\{u_l\}_l$ by E. First observe, by a compactness argument, that if no subsequence of $\{e_l\}_l$ dominates (resp. is dominated by) $\{u_l\}_l$ and if $\{A_m\}_m$ is a partition of $\mathbb{N}$ into finite sets, then for every number D there exists M such that whenever K is a subset of $\mathbb{N}$ such that $|K \cap A_m| = 1$ for $1 \leq m \leq M$ then $\|I\colon X|_K \to E|_K\| \geq D$ (resp. $\|I\colon E|_K \to X|_K\| \geq D$).

By passing to subsequences of $\{e_l\}_l$ we may assume that

(Ia) either $\|I\colon E|_L \to X|_L\| = \infty$ for all infinite subsequences L of $\mathbb{N}$,

(Ib) or $\|I\colon E \to X\| < \infty$.

Partition the whole basis $\{e_l\}_l$ into four infinite sets $\{e_{i,l}\}_l$ for $i = 1, \ldots, 4$. Of course E is the spreading model for each $\{e_{i,l}\}_l$ $(i = 1, \ldots, 4)$. In the proof we shall use a shorthand notation $\sim$ to denote that two sequences are 2-equivalent.

CASE (Ia): Set $f_{1,l} = e_{1,l}$ for all l and let $\Delta_1 = \{A_{1,m}\}_m$ consist of singletons. Let $F_1 = \overline{\text{span}}[f_{1,l}]_l$. Let $\Delta_2 = \{A_{2,m}\}_m$ be a partition of $\mathbb{N}$ into successive intervals such that $\|I\colon E|_{A_{2,m}} \to F_1|_{A_{2,m}}\| \geq 2^{2m}$ for all m. Define $f_{2,l}$ for $l \in A_{2,m}$ by induction in m. For $l \in A_{2,1}$ set $f_{2,l} = e_{2,s_1+l}$, where s_1 is so large that $\{f_{2,l}\}_{l\in A_{2,1}} \sim \{u_l\}_{l=1}^{|A_{2,1}|}$. Then having defined $f_{2,l}$ for all $l \in A_{2,m}$ set, for $l \in A_{2,m+1}$, $f_{2,l} = e_{2,s_{m+1}+l}$, where $s_{m+1} > s_m + |A_{2,m}|$ is so large that $\{f_{2,l}\}_{l\in A_{2,m+1}} \sim \{u_l\}_{l=1}^{|A_{2,m+1}|}$. In particular, $\{f_{2,l}\}_l$ is a subsequence of $\{e_{2,l}\}_l$. Let $F_2 = \overline{\text{span}}[f_{2,l}]_l$.

By our initial remark we can define by induction a sequence $0 = M_0 < M_1 < \cdots < M_j < \cdots$ such that for every $j = 1, 2, \ldots$ whenever $K \subset \mathbb{N}$ satisfies $|K \cap A_{2,m}| = 1$ for $M_{j-1} < m \leq M_j$, then $\|I\colon E|_K \to F_2|_K\| \geq 2^{2j}$. Then set $A_{3,j} = \bigcup_{m=M_{j-1}+1}^{M_j} A_{2,m}$ for $j = 1,2,\ldots$. Define $\{f_{3,l}\}_l$ as a subsequence of $\{e_{3,l}\}_l$ such that $\{f_{3,l}\}_{l\in A_{3,j}} \sim \{u_l\}_{l=1}^{|A_{3,j}|}$, and set $F_3 = \overline{\text{span}}[f_{3,l}]_l$.

Finally, using the same construction once more for the partition $\Delta_3 = \{A_{3,j}\}_j$, find a partition $\Delta_4 = \{A_{4,n}\}_n$ with $\Delta_4 \prec \Delta_3$ such that for every $K \in \mathcal{K}_{4,n}$ we have $\|I\colon E|_K \to F_3|_K\| \geq 2^{2n}$ (for $n = 1,2,\ldots$). Then define $\{f_{4,l}\}_l$ as a subsequence of $\{e_{4,l}\}_l$ such that $\{f_{4,l}\}_{l\in A_{4,n}} \sim \{u_l\}_{l=1}^{|A_{4,n}|}$ for $n = 1,2,\ldots$; and by $F_4 = \overline{\text{span}}[f_{4,l}]_l$ denote the corresponding subspace.

To check conditions (6), it is enough to observe that for $i = 1,2,3$ and $m = 1,2,\ldots$, and every $K \in \mathcal{K}_{i+1,m}$ we have $K \subset A_{i+1,m}$; and hence $\{f_{i+1,l}\}_{l\in K} \sim \{u_l\}_{l=1}^{|K|}$, in other words, $F_{i+1}|_K \sim E|_K$. Therefore, by our construction,

$$\|I\colon F_{i+1}|_K \to F_i|_K\| \geq (1/2)\|I\colon E|_K \to F_i|_K\| \geq 2^{2m-1} = \delta_m^{-1}.$$

To check (4), note that for $i = 1,2,3$ and $m = 1,2,\ldots$ and every $A_{i,m}$ one has $F_i|_{A_{i,m}} \sim F_{i+1}|_{A_{i,m}} \sim E|_{A_{i,m}}$.

Finally, all the F_s's are spanned by subsequences of the 1-unconditional basis, hence the projections $R_K^{(s)}$ have norms equal to 1. This shows case (Ia).

CASE (Ib): Set $f_{1,l} = e_{1,l}$ for all l and let $\{A_{1,m}\}_m$ be singletons. Let $F_1 = \overline{\text{span}}[f_{1,l}]_l$. Since $\|I\colon E \to F_1\| \leq \|I\colon E \to X\| < \infty$, we must have $\|I\colon F_1 \to E\| = \infty$. Let $\Delta_2 = \{A_{2,m}\}_m$ be a partition of $\mathbb{N}$ into successive intervals such that $\|I\colon F_1|_{A_{2,m}} \to E|_{A_{2,m}}\| \geq 2^{2m}$ for all m. Define the sequence $\{f_{2,l}\}_l$ as a subsequence of $\{e_{2,l}\}_l$ in the same way as in Case (Ia) and let F_2 be its span. Then for all m we have $F_2|_{A_{2,m}} \sim E|_{A_{2,m}}$, hence $\|I\colon F_1|_{A_{2,m}} \to F_2|_{A_{2,m}}\| \geq 2^{2m}$.

Since $\|I\colon E \to F_1\| < \infty$, and $\|I\colon E \to F_2\| < \infty$, then also

$$\|I\colon E \to D(F_1 \oplus F_2)\| < \infty, \qquad \text{where } D(F_1 \oplus F_2) = \overline{\text{span}}[f_{1,l} + f_{2,l}]_l.$$

Thus $\|I\colon D(F_1 \oplus F_2) \to E\| = \infty$. As in Case (Ia) we can find an increasing sequence $\{M_j\}$ of integers and a partition $\Delta_3 = \{A_{3,j}\}_j$ of $\mathbb{N}$ with $A_{3,j} = \bigcup_{m=M_{j-1}+1}^{M_j} A_{2,m}$ such that for every $K \in \mathcal{K}_{3,j}$ we have

$$\|I\colon D(F_1 \oplus F_2)|_K \to E|_K\| \geq 2^{2j}.$$

We then define a subsequence $\{f_{3,l}\}_l$ of $\{e_{3,l}\}_l$, so that its span F_3 satisfies $F_3|_{A_{3,j}} \sim E|_{A_{3,j}}$, for all j.

Finally we define a partition $\Delta_4 = \{A_{4,n}\}_n$ with $\Delta_4 \prec \Delta_3$ such that for every $K \in \mathcal{K}_{4,n}$ we have $\|I\colon D(F_1 \oplus F_2 \oplus F_3)|_K \to E|_K\| \geq 2^{2n}$ (for $n = 1,2,\ldots$), and then we define a subsequence $\{f_{4,l}\}_l$ of $\{e_{3,l}\}_l$, so that its span F_4 satisfies $F_4|_{A_{4,n}} \sim E|_{A_{4,n}}$, for all n.

Just as in Case (Ia), it is easy to check that (4) and (5) are satisfied, and that all the projections $R_K^{(s)}$ have norms equal to 1. ∎

Proof of Theorem 3.1 in Case (II): The key argument in [K-T] was contained in Proposition 3.3. Slight modifications of the statement of this proposition and of all the proofs in Section 3 of [K-T] work here, provided that we can use Theorem ⋆2.1. This requires proving a uniform bound for norms of natural projections on subspaces involved. This will follow by a deeper analysis of the construction in the case of subsymmetric bases. We formalize it by introducing a notion of a C-projection-regular pair which replaces the concept of a C-regular pair $\{\Delta, F\}$ from [K-T].

We say that a partition $\Delta = \{A_m\}_m$ of $\mathbb{N}$ into consecutive intervals and a space F with a normalized Schauder basis $\{f_l\}_l$ form a C-projection-regular pair, if the following conditions are satisfied:

(i) $\operatorname{equiv}\left(F|_{A_m}, l_2^{|A_m|}\right) \le C$ and the natural projection $Q_{A_m}\colon F \to F|_{A_m}$ has the norm $\|Q_{A_m}\| \le C$, for $m = 1, 2, \ldots$;

(ii) for every $L \in \mathcal{L}(\Delta)$, the basis $\{f_l\}_{l \in L}$ in $F|_L$ is 1-unconditional and the natural projection $Q_L\colon F \to F|_L$ has the norm $\|Q_L\| \le C$ (here $\mathcal{L}(\Delta)$ is the family of all $L \subset \mathbb{N}$ such that $|L \cap A_m| = 1$ for all m);

(iii) for arbitrary $L, L' \in \mathcal{L}(\Delta)$ one has $\operatorname{equiv}\left(F|_L, F|_{L'}\right) = 1$.

The modified Proposition 3.3 now additionally assumes that all spaces E_i have subsymmetric bases, and it asserts that the constructed pairs $\{\Delta_i, F_i\}$ are C-projection-regular.

The use of Rademacher functions in a setting of discrete Banach lattices gives more information for spaces with subsymmetric bases. For example, Tzafriri's argument mentioned in [K-T] uses this approach; this can be also tackled by a modification of inequalities from Lemma 3.2 in [K-T]. One gets that for $m \in \mathbb{N}$ and $N = 2^m$, if E is an N-dimensional space of cotype $r < \infty$ with a 1-subsymmetric basis $\{e_j\}_j$, then m Rademacher vectors $\{f_l\}$ of the form

$$f_l = \alpha \sum_{j=1}^{N} \varepsilon_l(j) e_j \tag{7}$$

are C-equivalent to the unit vector basis in ℓ_2^m, where C depends on r and the cotype r constant. Moreover, $\{\varepsilon_l(j)\}$ are values of Rademacher functions on the diadic partition of the interval $[0, 1]$, and so they are mutually orthogonal, that is, $\sum_j \varepsilon_l(j)\varepsilon_{l'}(j) = 0$ if $l \neq l'$.

For a given space E of cotype r with a subsymmetric basis $\{e_j\}_j$, and a partition $\Delta = \{A_m\}_m$ of $\mathbb{N}$ into consecutive intervals, we repeat word by word the first part of the proof of Proposition 3.3 in [K-T] to get a subspace $F = \overline{\mathrm{span}}[f_l]_l$ of the following form: we fix arbitrary successive intervals of positive integers, $\{J_m\}$, with $|J_m| = 2^{|A_m|}$ and by formula (7) we define vectors f_l by

$$f_l = \alpha_m \sum_{j \in J_m} \varepsilon_l(j) e_j \qquad \text{for } l \in A_m,\ m = 1, 2, \ldots.$$

All the conditions from [K-T] are satisfied as before; we need only check the boundness of the natural projections, $Q_K\colon F \to F|_K$ given by $Q_K(\sum_l a_l f_l) = \sum_{l \in K} a_l f_l$, for $K \subset \mathbb{N}$.

In condition (i), Q_{A_m} is equal to the restriction to F of the natural projection P_{J_m} in E onto coordinates from J_m, hence is obviously bounded.

In condition (ii), let $L = \{l_m\}_m \in \mathcal{L}(\Delta)$. Then $\{f_{l_m}\}_m$ form a subspace $F|_L$ of E spanned by constant coefficient successive blocks. Consider the averaging projection P_L in E given by the formula

$$P_L\Big(\sum_j a_j e_j\Big) = \sum_m (\alpha_m |J_m|)^{-1} \Big(\sum_{j \in J_m} \varepsilon_{l_m}(j) a_j\Big) f_{l_m}.$$

The orthogonality relations of sign vectors $\{\varepsilon_l(j)\}$ imply that for any $l \notin L$ we have $P_L(f_l) = 0$. So the projection Q_L acting on F is equal to the restriction of P_L to F. Finally let us recall ([L-T] 3.a.4), that averaging projections in a space with a subsymmetric basis have the norm $\|P_L\| \le 2$, and this shows that $\|Q_L\| \le 2$.

Now the proof of Theorem 3.1 starts by writing a space X with a subsymmetric basis as an unconditional sum of 13 spaces E_i with subsymmetric bases. Then the modified version of Proposition 3.3 proved above allows us to use Theorem $\star$2.1 in the original argument. ∎

Remark: Actually, even in Case II, a subspace without l.u.st. can be constructed which has 2-dimensional unconditional decomposition. This can be done by an additional careful use of Krivine's theorem on finite block representability of ℓ_p. More details on this can be found in a recent survey paper [T]. However, the construction of Case II based on [K-T] is of interest in other situations than Theorem 3.1 as well. Similarly, Theorem $\star$2.1 is, as far as we know, one of very few results of this type which do not assume the unconditionality of the whole Schauder decomposition, but just of its subsets. The details will apear elsewhere.

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